CHOW GROUPS AND DERIVED CATEGORIES OF K3 SURFACES

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ABSTRACT. The geometry of a K3 surface (over $\mathbb{C}$ or over $\bar{\mathbb{Q}}$) is reflected by its Chow group and its bounded derived category of coherent sheaves in different ways. The Chow group can be infinite dimensional over $\mathbb{C}$ (Mumford) and is expected to inject into cohomology over $\bar{\mathbb{Q}}$ (Bloch–Beilinson). The derived category is difficult to describe explicitly, but its group of autoequivalences can be studied by means of the natural representation on cohomology. Conjecturally (Bridgeland) the kernel of this representation is generated by squares of spherical twists. The action of these spherical twists on the Chow ring can be determined explicitly by relating it to the natural subring introduced by Beauville and Voisin.

1. INTRODUCTION

In algebraic geometry a K3 surface is a smooth projective surface X over a fixed field K with trivial canonical bundle $\omega_X \simeq \Omega_X^2$ and $H^1(X, \mathcal{O}_X) = 0$. For us the field K will be either a number field, the field of algebraic numbers $\bar{\mathbb{Q}}$ or the complex number field $\mathbb{C}$. Non-projective K3 surfaces play a central role in the theory of K3 surfaces and for some of the results that will be discussed in this text in particular, but here we will not discuss those more analytical aspects.

An explicit example of a K3 surface is provided by the Fermat quartic in $\mathbb{P}^3$ given as the zero set of the polynomial $x_0^4 + \ldots + x_3^4$. Kummer surfaces, i.e. minimal resolutions of the quotient of abelian surfaces by the sign involution, and elliptic K3 surfaces form other important classes of examples. Most of the results and questions that will be mentioned do not loose any of their interest when considered for one of theses classes of examples or any other particular K3 surface.

This text deals with three objects naturally associated with any K3 surface X:

$$\mathrm{D}^\mathrm{b}(X), \ \mathrm{CH}^*(X) \text{ and } H^*(X, \mathbb{Z}).$$

If X is defined over $\mathbb{C}$, its *singular cohomology* $H^*(X, \mathbb{Z})$ is endowed with the intersection pairing and a natural Hodge structure. The *Chow group* $\mathrm{CH}^*(X)$ of X, defined over an arbitrary field, is a graded ring that encodes much of the algebraic geometry. The *bounded derived category* $\mathrm{D}^\mathrm{b}(X)$, a linear triangulated category, is a more complicated invariant and in general difficult to control.

As we will see, all three objects, $H^*(X, \mathbb{Z})$, $\mathrm{CH}^*(X)$, and $\mathrm{D}^\mathrm{b}(X)$ are related to each other. On the one hand, $\widetilde{H}(X, \mathbb{Z})$ as the easiest of the three can be used to capture some of the features of the other two. But on the other hand and maybe a little surprising, one can deduce from the

more rigid structure of $\mathrm{D}^b(X)$ as a linear triangulated category interesting information about cycles on X, i.e. about some aspects of $\mathrm{CH}^*(X)$.

This text is based on my talk at the conference 'Classical algebraic geometry today' at the MSRI in January 2009 and is meant as a non-technical introduction to the standard techniques in the area. At the same time it surveys recent developments and presents some new results on a question on symplectomorphisms that was raised in this talk (see Section 6). I wish to thank the organizers for the invitation to a very stimulating conference.

2. Cohomology of K3 surfaces

The second singular cohomology of a complex K3 surface is endowed with the additional structure of a weight two Hodge structure and the intersection pairing. The Global Torelli theorem shows that it determines the K3 surface uniquely. We briefly recall the main features of this Hodge structure and of its extension to the Mukai lattice which governs the derived category of the K3 surface. For the general theory of complex K3 surfaces see e.g. [1] or [3]. In this section all K3 surfaces are defined over $\mathbb{C}$.

2.1. To any complex K3 surface X we can associate the singular cohomology $H^*(X, \mathbb{Z})$ (of the underlying complex or topological manifold). Clearly, $H^0(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$. Hodge decomposition yields $H^1(X, \mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X) = 0$, since by assumption $H^{0,1}(X) \simeq H^1(X, \mathcal{O}_X) = 0$, and hence $H^1(X, \mathbb{Z}) = 0$. One can also show $H^3(X, \mathbb{Z}) = 0$. Thus, the only interesting cohomology group is $H^2(X, \mathbb{Z})$ which together with the intersection pairing is abstractly isomorphic to the unique even unimodular lattice of signature $(3, 19)$ given by $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Here, U is the hyperbolic plane and $E_8(-1)$ is the standard root lattice E_8 changed by a sign. Thus, the full cohomology $H^*(X, \mathbb{Z})$ endowed with the intersection pairing is isomorphic to $U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$.

For later use we introduce $\widetilde{H}(X, \mathbb{Z})$, which denotes $H^*(X, \mathbb{Z})$ with the Mukai paring, i.e. with a sign change in the pairing between H^0 and H^4. Note that as abstract lattices $H^*(X, \mathbb{Z})$ and $\widetilde{H}(X, \mathbb{Z})$ are isomorphic.

2.2. The complex structure of the K3 surface X induces a weight two Hodge structure on $H^2(X, \mathbb{Z})$ given explicitly by the decomposition $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$. It is determined by the complex line $H^{2,0}(X) \subset H^2(X, \mathbb{C})$ which is spanned by a trivializing section of ω_X and by requiring the decomposition to be orthogonal with respect to the intersection pairing. This natural Hodge structure induces at the same time a weight two Hodge structure on the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$ by setting $\widetilde{H}^{2,0}(X) = H^{2,0}(X)$ and requiring $(H^0 \oplus H^4)(X, \mathbb{C}) \subset \widetilde{H}^{1,1}(X)$.

The Global Torelli theorem and its derived version, due to Piatetski-Shapiro and Shafarevich resp. Mukai and Orlov, can be stated as follows. For complex projective K3 surfaces X and X' one has:

i) There exists an isomorphism $X \simeq X'$ (over $\mathbb{C}$) if and only if there exists an isometry of Hodge structures $H^2(X, \mathbb{Z}) \simeq H^2(X', \mathbb{Z})$.

ii) There exists a $\mathbb{C}$-linear exact equivalence $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X')$ if and only if there exists an isometry of Hodge structures $\widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(X', \mathbb{Z})$.

Note that for purely lattice theoretical reasons the weight two Hodge structures $\widetilde{H}(X, \mathbb{Z})$ and $\widetilde{H}(X', \mathbb{Z})$ are isometric if and only if their transcendental parts (see 2.3) are.

2.3. The Hodge index theorem shows that the intersection pairing on $H^{1,1}(X, \mathbb{R})$ has signature $(1, 19)$. Thus the cone of classes α with $\alpha^2 > 0$ decomposes into two connected components. The connected component $\mathcal{C}_X$ that contains the Kähler cone $\mathcal{K}_X$, i.e. the cone of all Kähler classes, is called the positive cone. Note that for the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$ the set of real $(1, 1)$-classes of positive square is connected.

The Néron–Severi group $\mathrm{NS}(X)$ is identified with $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ and its rank is the Picard number $\rho(X)$. Since X is projective, the intersection form on $\mathrm{NS}(X)_{\mathbb{R}}$ has signature $(1, \rho(X) - 1)$. The transcendental lattice $T(X)$ is by definition the orthogonal complement of $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$. Hence, $H^2(X, \mathbb{Q}) = \mathrm{NS}(X)_{\mathbb{Q}} \oplus T(X)_{\mathbb{Q}}$ which can be read as an orthogonal decomposition of weight two rational Hodge structures (but in general not over $\mathbb{Z}$). Note that $T(X)_{\mathbb{Q}}$ cannot be decomposed further, it is an irreducible Hodge structure. The ample cone is the intersection of the Kähler cone $\mathcal{K}_X$ with $\mathrm{NS}(X)_{\mathbb{R}}$ and is spanned by ample line bundles.

Analogously, one has the extended Néron–Severi group

$$\widetilde{\mathrm{NS}}(X) := \widetilde{H}^{1,1}(X) \cap \widetilde{H}(X, \mathbb{Z}) = \mathrm{NS}(X) \oplus (H^0 \oplus H^4)(X, \mathbb{Z}).$$

Note that $\widetilde{\mathrm{NS}}(X)$ is simply the lattice of all algebraic classes. More precisely, $\widetilde{\mathrm{NS}}(X)$ can be seen as the image of the cycle map $\mathrm{CH}^*(X) \longrightarrow H^*(X, \mathbb{Z})$ or the set of all Mukai vectors $v(E) = \mathrm{ch}(\mathrm{E}).\sqrt{\mathrm{td}(\mathrm{X})} = \mathrm{ch}(\mathrm{E}).(1, 0, 1)$ with $E \in \mathrm{D}^{\mathrm{b}}(X)$. Note that the transcendental lattice in $\widetilde{H}(X, \mathbb{Z})$ coincides with $T(X)$.

2.4. The so-called (-2)-classes, i.e. integral $(1, 1)$-classes δ with $\delta^2 = -2$, play a central role in the classical theory as well as in the modern part related to derived categories and Chow groups.

Classically, one considers the set Δ_X of (-2)-classes in $\mathrm{NS}(X)$. E.g. every smooth rational curve $\mathbb{P}^1 \simeq C \subset X$ defines by adjunction a (-2)-class, hence C is called a (-2)-curve. Examples of (-2)-classes in the extended Néron–Severi lattice $\widetilde{\mathrm{NS}}(X)$ are provided by the Mukai vector $v(E)$ of spherical objects $E \in \mathrm{D}^{\mathrm{b}}(X)$ (see 4.2 and 5.1). Note that $v(\mathcal{O}_C) \neq [C]$, but $v(\mathcal{O}_C(-1)) = [C]$. For later use we introduce $\widetilde{\Delta}_X$ as the set of (-2)-classes in $\widetilde{\mathrm{NS}}(X)$.

Clearly, an ample or, more generally, a Kähler class has positive intersection with all effective curves and with (-2)-curves in particular. Conversely, one knows that every class $\alpha \in \mathcal{C}_X$ with $(\alpha.C) > 0$ for all (-2)-curves is a Kähler class (cf. [1]).

To any (-2)-class δ one associates the reflection $s_\delta : \alpha \longmapsto \alpha + (\alpha.\delta)\delta$ which is an orthogonal transformation of the lattice also preserving the Hodge structure. The Weyl group is by definition the subgroup of the orthogonal group generated by reflections s_δ. So one has two groups $W_X \subset \mathrm{O}(H^2(X, \mathbb{Z}))$ and $\widetilde{W}_X \subset \mathrm{O}(\widetilde{H}(X, \mathbb{Z}))$.

The union of hyperplanes $\bigcup_{\delta \in \Delta_X} \delta^\perp$ is locally finite in the interior of $\mathcal{C}_X$ and endows $\mathcal{C}_X$ with a chamber structure. The Weyl group W_X acts simply transitively on the set of chambers and

the Kähler cone is one of the chambers. The action of W_X on $\mathrm{NS}(X)_\mathbb{R} \cap \mathcal{C}_X$ can be studied analogously. It can also be shown that reflections $s_{[C]}$ with $C \subset X$ smooth rational curves generate W_X.

Another part of the Global Torelli theorem complementing i) in 2.2 says that a non-trivial automorphism $f \in \mathrm{Aut}(X)$ acts always non-trivially on $H^2(X, \mathbb{Z})$. Moreover, any Hodge isometry of $H^2(X, \mathbb{Z})$ preserving the positive cone is induced by an automorphism up to the action of W_X. In fact, Piatetski-Shapiro and Shafarevich also showed that the action on $\mathrm{NS}(X)$ is essentially enough to determine f. More precisely, one knows that the natural homomorphism

$$\mathrm{Aut}(X) \longrightarrow \mathrm{O}(\mathrm{NS}(X))/W_X$$

has finite kernel and cokernel. Roughly, the kernel is finite because an automorphism that leaves invariant a polarization is an isometry of the underlying hyperkähler structure and these isometries form a compact group. For the finiteness of the cokernel note that some high power of any automorphism f always acts trivially on $T(X)$.

The extended Néron–Severi group plays also the role of a period domain for the space of stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ (see 4.4). For this consider the open set $\mathcal{P}(X) \subset \widetilde{\mathrm{NS}}(X)_\mathbb{C}$ of vectors whose real and imaginary parts span a positively oriented positive plane. Then let $\mathcal{P}_0(X) \subset \mathcal{P}(X)$ be the complement of the union of all codimension two sets $\delta^\perp$ with $\delta \in \widetilde{\mathrm{NS}}(X)$ and $\delta^2 = -2$ (or, equivalently, $\delta = v(E)$ for some spherical object $E \in \mathrm{D}^{\mathrm{b}}(X)$ as we will explain later):

$$\mathcal{P}_0(X) := \mathcal{P}(X) \setminus \bigcup_{\delta \in \widetilde{\Delta}_X} \delta^\perp.$$

Since the signature of the intersection form on $\widetilde{\mathrm{NS}}(X)$ is $(2, \rho(X))$, the set $\mathcal{P}_0(X)$ is connected. Its fundamental group $\pi_1(\mathcal{P}_0(X))$ is generated by loops around each $\delta^\perp$ and the one induced by the natural $\mathbb{C}^*$-action.

3. Chow ring

We now turn to the second object that can naturally be associated with any K3 surface X defined over an arbitrary field K, the Chow group $\mathrm{CH}^*(X)$. For a separably closed field like $\bar{\mathbb{Q}}$ or $\mathbb{C}$ it is torsion free due to a theorem of Roitman [30] and for number fields we will simply ignore everything that is related to the possible occurrence of torsion. The standard reference for Chow groups is Fulton's book [9]. For the interplay between Hodge theory and Chow groups see e.g. [31].

3.1. The Chow group $\mathrm{CH}^*(X)$ of a K3 surface (over K) is the group of cycles modulo rational equivalence. Thus, $\mathrm{CH}^0(X) \simeq \mathbb{Z}$ (generated by $[X]$) and $\mathrm{CH}^1(X) = \mathrm{Pic}(X)$. The interesting part is $\mathrm{CH}^2(X)$ which behaves differently for $K = \bar{\mathbb{Q}}$ and $K = \mathbb{C}$. Let us begin with the following celebrated result of Mumford [27].

Theorem 3.1. (Mumford) *If $K = \mathbb{C}$, then $\mathrm{CH}^2(X)$ is infinite dimensional.*

(A priori $\text{CH}^2(X)$ is simply a group, so one needs to explain what it means that $\text{CH}^2(X)$ is infinite dimensional. A first very weak version says that $\dim_{\mathbb{Q}} \text{CH}^2(X)_{\mathbb{Q}} = \infty$. For a more geometrical and more precise definition of infinite dimensionality see e.g. [31].)

For $K = \bar{\mathbb{Q}}$ the situations is expected to be different. The Bloch–Beilinson conjectures lead one to the following conjecture for K3 surfaces.

Conjecture 3.2. *If K is a number field or $K = \bar{\mathbb{Q}}$, then $\text{CH}^2(X)_{\mathbb{Q}} = \mathbb{Q}$.*

So, if X is a K3 surface defined over $\bar{\mathbb{Q}}$, then one expects $\dim_{\mathbb{Q}} \text{CH}^2(X)_{\mathbb{Q}} = 1$, whereas for the complex K3 surface $X_{\mathbb{C}}$ obtained by base change from X one knows $\dim_{\mathbb{Q}} \text{CH}^2(X_{\mathbb{C}})_{\mathbb{Q}} = \infty$. To the best of my knowledge not a single example of a K3 surface X defined over $\bar{\mathbb{Q}}$ is known where finite dimensionality of $\text{CH}^2(X)_{\mathbb{Q}}$ could be verified.

Also note that the Picard group does not change under base change from $\bar{\mathbb{Q}}$ to $\mathbb{C}$, i.e. for X defined over $\bar{\mathbb{Q}}$ one has $\text{Pic}(X) \simeq \text{Pic}(X_{\mathbb{C}})$ (see 5.3). But over the actual field of definition of X, which is a number field in this case, the Picard group can be strictly smaller.

The central argument in Mumford's proof is that an irreducible component of the closed subset of effective cycles in X^n rationally equivalent to a given cycle must be proper, due to the existence of a non-trivial regular two-form on X, and that a countable union of those cannot cover X^n if the base field is not countable. This idea was later formalized and has led to many more results proving non-triviality of cycles under non-vanishing hypotheses on the non-algebraic part of the cohomology (see e.g. [31]). There is also a more arithmetic approach to produce arbitrarily many non-trivial classes in $\text{CH}^2(X)$ for a complex K3 surface X which proceeds via curves over finitely generated field extensions of $\bar{\mathbb{Q}}$ and embeddings of their function fields into $\mathbb{C}$. See e.g. [13].

The degree of a cycle induces a homomorphism $\text{CH}^2(X) \longrightarrow \mathbb{Z}$ and its kernel $\text{CH}^2(X)_0$ is the group of homologically (or algebraically) trivial classes. Thus, the Bloch–Beilinson conjecture for a K3 surface X over $\bar{\mathbb{Q}}$ says that $\text{CH}^2(X)_0 = 0$ or, equivalently, that

$$\text{CH}^*(X) \simeq \widetilde{\text{NS}}(X_{\mathbb{C}}) \lhook\joinrel\longrightarrow \widetilde{H}(X_{\mathbb{C}}, \mathbb{Z}).$$

3.2.   The main results presented in my talk were triggered by the paper of Beauville and Voisin [4] on a certain natural subring of $\text{CH}^*(X)$. They show in particular that for a complex K3 surface X there is a natural class $c_X \in \text{CH}^2(X)$ of degree one with the following properties:

i) $c_X = [x]$ for any point $x \in X$ contained in a (singular) rational curve $C \subset X$.
ii) $c_1(L)^2 \in \mathbb{Z}c_X$ for any $L \in \text{Pic}(X)$.
iii) $c_2(X) = 24c_X$.

Let us introduce

$$R(X) := \text{CH}^0(X) \oplus \text{CH}^1(X) \oplus \mathbb{Z}c_X.$$

Then ii) shows that $R(X)$ is a subring of $\text{CH}^*(X)$. A different way of expressing ii) and iii) together is to say that for any $L \in \text{Pic}(X)$ the Mukai vector $v^{\text{CH}}(L) = \text{ch}(L)\sqrt{\text{td}(X)}$ is contained in $R(X)$ (see 4.1). It will be in this form that the results of Beauville and Voisin can be generalized in a very natural form to the derived context (Theorem 5.1).

Note that the cycle map induces an isomorphism $R(X) \simeq \widetilde{\mathrm{NS}}(X)$ and that for a K3 surface X over $\bar{\mathbb{Q}}$ the Bloch–Beilinson conjecture can be expressed by saying that base change yields an isomorphism $\mathrm{CH}^*(X) \simeq R(X_{\mathbb{C}})$.

So, the natural filtration $\mathrm{CH}^*(X)_0 \subset \mathrm{CH}^*(X)$ (see also below) with quotient $\widetilde{\mathrm{NS}}(X)$ admits a split given by $R(X)$. This can be written as $\mathrm{CH}^*(X) = R(X) \oplus \mathrm{CH}^*(X)_0$ and seems to be a special feature of K3 surfaces and higher-dimensional symplectic varieties. E.g. in [5] it was conjectured that any relation between $c_1(L_i)$ of line bundles L_i on an irreducible symplectic variety X in $H^*(X)$ also holds in $\mathrm{CH}^*(X)$. The conjecture was completed to also incorporate Chern classes of X and proved for low-dimensional Hilbert schemes of K3 surfaces by Voisin in [32]. See also the more recent thesis by Ferretti [8] which deals with double EPW sextics, which are special deformations four-dimensional Hilbert schemes.

3.3.   The Bloch–Beilinson conjectures also predict for smooth projective varieties X the existence of a functorial filtration

$$0 = F^{p+1}\mathrm{CH}^p(X) \subset F^p\mathrm{CH}^p(X) \subset \ldots \subset F^1\mathrm{CH}^p(X) \subset F^0\mathrm{CH}^p(X)$$

whose first step F^1 is simply the kernel of the cycle map. Natural candidates for such a filtration were studied e.g. by Green, Griffiths, Jannsen, Lewis, Murre, and S. Saito (see [12] and the references therein).

For a surface X the interesting part of this filtration is $0 \subset \ker(\mathrm{alb}_X) \subset \mathrm{CH}^2(X)_0 \subset \mathrm{CH}^2(X)$. Here $\mathrm{alb}_X : \mathrm{CH}^2(X)_0 \longrightarrow \mathrm{Alb}(X)$ denotes the Albanese map.

A cycle $\Gamma \in \mathrm{CH}^2(X \times X)$ naturally acts on cohomology and on the Chow group. We write $[\Gamma]_*^{i,0}$ for the induced endomorphism of $H^0(X, \Omega_X^i)$ and $[\Gamma]_*$ for the action on $\mathrm{CH}^2(X)$. The latter respects the natural filtration $\ker(\mathrm{alb}_X) \subset \mathrm{CH}^2(X)_0 \subset \mathrm{CH}^2(X)$ and thus induces an endomorphism $\mathrm{gr}[\Gamma]_*$ of the graded object $\ker(\mathrm{alb}_X) \oplus \mathrm{Alb}(X) \oplus \mathbb{Z}$.

The following is also a consequence of Bloch's conjecture, see [2] or [31, Ch. 11], not completely unrelated to Conjecture 3.2.

Conjecture 3.3. $[\Gamma]_*^{2,0} = 0$ *if and only if* $\mathrm{gr}[\Gamma]_* = 0$ *on* $\ker(\mathrm{alb}_X)$.

It is known that this conjecture is implied by the Bloch–Beilinson conjecture for $X \times X$ when X and Γ are defined over $\bar{\mathbb{Q}}$. But otherwise, very little is known about it. Note that the analogous statement $[\Gamma]_*^{1,0} = 0$ if and only if $\mathrm{gr}[\Gamma]_* = 0$ on $\mathrm{Alb}(X)$ holds true by definition of the Albanese.

For K3 surfaces the Albanese map is trivial and so the Bloch–Beilinson filtration for K3 surfaces is simply $0 \subset \ker(\mathrm{alb}_X) = \mathrm{CH}^2(X)_0 \subset \mathrm{CH}^2(X)$. In particular Conjecture 3.3 for a K3 surface becomes: $[\Gamma]_*^{2,0} = 0$ if and only if $\mathrm{gr}[\Gamma]_* = 0$ on $\mathrm{CH}^2(X)_0$. In this form the conjecture seems out of reach for the time being, but the following special case seems more accessible and we will explain in Section 6 to what extend derived techniques can be useful to answer it.

Conjecture 3.4. *Let* $f \in \mathrm{Aut}(X)$ *be a symplectomorphism of a complex projective K3 surface* X, *i.e.* $f^* = \mathrm{id}$ *on* $H^{2,0}(X)$. *Then* $f^* = \mathrm{id}$ *on* $\mathrm{CH}^2(X)$.

Remark 3.5. Note that the converse is true: If $f \in \mathrm{Aut}(X)$ acts as id on $\mathrm{CH}^2(X)$, then f is a symplectomorphism. This is reminiscent of a consequence of the Global Torelli theorem which for a complex projective K3 surface X states:

- $f = \mathrm{id}$ if and only if $f^* = \mathrm{id}$ on the Chow ring(!) $\mathrm{CH}^*(X)$.

4. DERIVED CATEGORY

The Chow group $\mathrm{CH}^*(X)$ is the space of cycles divided by rational equivalence. Equivalently, one could take the abelian or derived category of coherent sheaves on X and pass to the Grothendieck K-groups. It turns out that considering the more rigid structure of a category that lies behind the Chow group can lead to new insight. See [15] for a general introduction to derived categories and for more references to the original literature.

4.1. For a K3 surface X over a field K the category $\mathrm{Coh}(X)$ of coherent sheaves on X is a K-linear abelian category and its *bounded derived category*, denoted $\mathrm{D}^b(X)$, is a K-linear triangulated category.

If $E^\bullet$ is an object of $\mathrm{D}^b(X)$, its *Mukai vector* $v(E^\bullet) = \sum(-1)^i v(E^i) = \sum(-1)^i v(\mathcal{H}^i(E^\bullet)) \in \widetilde{\mathrm{NS}}(X) \subset \widetilde{H}(X, \mathbb{Z})$ is well defined. By abuse of notation, we will write the Mukai vector as a map
$$v : \mathrm{D}^b(X) \longrightarrow \widetilde{\mathrm{NS}}(X).$$
Since the Chern character of a coherent sheaf and the Todd genus of X exist as classes in $\mathrm{CH}^*(X)$, the Mukai vector with values in $\mathrm{CH}^*(X)$ can also be defined. This will be written as
$$v^{\mathrm{CH}} : \mathrm{D}^b(X) \longrightarrow \mathrm{CH}^*(X).$$
(It is a special feature of K3 surfaces that the Chern character really is integral.)

Note that $\mathrm{CH}^*(X)$ can also be understood as the Grothendieck K-group of the abelian category $\mathrm{Coh}(X)$ or of the triangulated category $\mathrm{D}^b(X)$, i.e. $K(X) \simeq K(\mathrm{Coh}(X)) \simeq K(\mathrm{D}^b(X)) \simeq \mathrm{CH}^*(X)$. (In order to exclude any torsion phenomena we assume here that K is algebraically closed, i.e. $K = \mathbb{C}$ or $K = \bar{\mathbb{Q}}$, or, alternatively, pass to the associated $\mathbb{Q}$-vector spaces.)

Clearly, the lift of a class in $\mathrm{CH}^*(X)$ to an object in $\mathrm{D}^b(X)$ is never unique. Of course, for certain classes there are natural choices, e.g. $v^{\mathrm{CH}}(L)$ naturally lifts to L which is a spherical object (see below).

4.2. Due to a result of Orlov, every K-linear equivalence $\Phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(X')$ between the derived categories of two smooth projective varieties is a Fourier–Mukai transform, i.e. there exists a unique object $\mathcal{E} \in \mathrm{D}^b(X \times X')$ such that Φ is isomorphic to the functor $\Phi_{\mathcal{E}} = p_*(q^*(\ ) \otimes \mathcal{E})$. Here p_*, q^*, and $\otimes$ are derived functors. It is known that if X is a K3 surface also X' is one.

It would be very interesting to use Orlov's result to deduce the existence of objects in $\mathrm{D}^b(X \times X')$ that are otherwise difficult to describe. However, we are not aware of any non-trivial example of a functor that can be shown to be an equivalence, or even just fully faithful, without actually describing it as a Fourier–Mukai transform.

Here is a list of essentially all known (auto)equivalences for K3 surfaces:

i) Any isomorphism $f : X \xrightarrow{\sim} X'$ induces an exact equivalence $f_* : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X')$ with Fourier–Mukai kernel the structure sheaf $\mathcal{O}_{\Gamma_f}$ of the graph $\Gamma_f \subset X \times X'$ of f.

ii) The tensor product $L \otimes (\ )$ for a line bundle $L \in \mathrm{Pic}(X)$ defines an autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ with Fourier–Mukai kernel $\Delta_* L$.

iii) An object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is called *spherical* if $\mathrm{Ext}^*(E, E) \simeq H^*(S^2, K)$ as graded vector spaces. The *spherical twist*

$$T_E : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$$

associated with it is the Fourier–Mukai equivalence whose kernel is given as the cone of the trace map $E^* \boxtimes E \longrightarrow (E^* \boxtimes E)|_\Delta \xrightarrow{\sim} \Delta_*(E^* \otimes E) \longrightarrow \mathcal{O}_\Delta$. (For example of spherical objects see 5.1.)

iv) If X' is a fine projective moduli space of stable sheaves and $\dim(X') = 2$, then the universal family $\mathcal{E}$ on $X \times X'$ (unique up to a twist with a line bundle on X') can be taken as the kernel of an equivalence $\mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X')$.

4.3.   Writing an equivalence as a Fourier–Mukai transform allows one to associate directly to any autoequivalence $\Phi : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$ of a complex K3 surface X an isomorphism

$$\Phi^H : \widetilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X, \mathbb{Z})$$

which in terms of the Fourier–Mukai kernel $\mathcal{E}$ is given by $\alpha \longmapsto p_*(q^*\alpha.v(\mathcal{E}))$. As was observed by Mukai, this isomorphism is defined over $\mathbb{Z}$ and not only over $\mathbb{Q}$. Moreover, it preserves the Mukai pairing and the natural weight two Hodge structure, i.e. it is an integral Hodge isometry of $\widetilde{H}(X, \mathbb{Z})$. As above, $v(\mathcal{E})$ denotes the Mukai vector $v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}(X \times X)}$.

Clearly, the latter makes also sense in $\mathrm{CH}^*(X \times X)$ and so one can as well associate to the equivalence Φ a group automorphism

$$\Phi^{\mathrm{CH}} : \mathrm{CH}^*(X) \xrightarrow{\sim} \mathrm{CH}^*(X).$$

The reason why the usual Chern character is replaced by the Mukai vector is the Grothendieck–Riemann–Roch formula. With this definition of Φ^H and Φ^{CH} one finds that $\Phi^H(v(E)) = v(\Phi(E))$ and $\Phi^{\mathrm{CH}}(v^{\mathrm{CH}}(E)) = v^{\mathrm{CH}}(\Phi(E))$ for all $E \in \mathrm{D}^{\mathrm{b}}(X)$.

Note that Φ^H and Φ^{CH} do not preserve, in general, neither the multiplicative structure nor the grading of $\widetilde{H}(X, \mathbb{Z})$ resp. $\mathrm{CH}^*(X)$.

The derived category $\mathrm{D}^{\mathrm{b}}(X)$ is difficult to describe in concrete terms. Its group of autoequivalences, however, seems more accessible. So let $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ denote the group of all K-linear exact equivalences $\Phi : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$ up to isomorphism. Then $\Phi \longmapsto \Phi^H$ and $\Phi \longmapsto \Phi^{\mathrm{CH}}$ define the two representations

$$\rho^H : \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{O}(\widetilde{H}(X, \mathbb{Z})) \qquad \text{and} \qquad \rho^{\mathrm{CH}} : \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}(\mathrm{CH}^*(X)).$$

Here, $\mathrm{O}(\widetilde{H}(X, \mathbb{Z}))$ is the group of all integral Hodge isometries of the weight two Hodge structure defined on the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$ and $\mathrm{Aut}(\mathrm{CH}^*(X))$ denotes simply the group of all automorphisms of the additive group $\mathrm{CH}^*(X)$.

Although $\mathrm{CH}^*(X)$ is a much bigger group than $\widetilde{H}(X, \mathbb{Z})$, at least over $K = \mathbb{C}$, both representations carry essentially the same information. More precisely one can prove (see [18]):

Theorem 4.1. $\ker(\rho^H) = \ker(\rho^{\mathrm{CH}})$.

In the following we will explain what is known about this kernel and the images of the representations ρ^H and ρ^{CH}.

4.4. Due to the existence of the many spherical objects in $\mathrm{D}^{\mathrm{b}}(X)$ and their associated spherical twists, the kernel $\ker(\rho^H) = \ker(\rho^{\mathrm{CH}})$ has a rather intriguing structure. Let us be a bit more precise: If $E \in \mathrm{D}^{\mathrm{b}}(X)$ is spherical, then T_E^H is the reflection s_δ in the hyperplane orthogonal to $\delta := v(E)$. Hence, the square T_E^2 is an element in $\ker(\rho^H)$ which is easily shown to be non-trivial.

Due to the existence of the many spherical objects on any K3 surface, e.g. all line bundles are spherical, and the complicated relations between them, the group generated by all T_E^2 is a very interesting object. In fact, conjecturally $\ker(\rho^H)$ is generated by the T_E^2's and the double shift. This and the expected relations between the spherical twists are expressed by the following conjecture of Bridgeland [6].

Conjecture 4.2. $\ker(\rho^H) = \ker(\rho^{\mathrm{CH}}) \simeq \pi_1(\mathcal{P}_0(X))$.

For the definition of $\mathcal{P}_0(X)$ see 2.4. The fundamental group of $\mathcal{P}_0(X)$ is generated by loops around each $\delta^\perp$ and the generator of $\pi_1(\mathcal{P}(X)) \simeq \mathbb{Z}$. The latter is naturally lifted to the autoequivalence given by the double shift $E \longmapsto E[2]$.

Since each (-2)-vector δ can be written as $\delta = v(E)$ for some spherical object, one can lift the loop around $\delta^\perp$ to T_E^2. However, the spherical object E is by no means unique. Just choose any other spherical object F and consider $T_F^2(E)$ which has the same Mukai vector as E. Even for a Mukai vector $v = (r, \ell, s)$ with $r > 0$ there is in general more than one spherical bundle(!) E with $v(E) = v$ (see 5.1).

Nevertheless, Bridgeland does construct a group homomorphism

$$\pi_1(\mathcal{P}_0(X)) \longrightarrow \ker(\rho^H) \subset \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)).$$

The injectivity of this map is equivalent to the simply connectedness of the distinguished component $\Sigma(X) \subset \mathrm{Stab}(X)$ of stability conditions considered by Bridgeland. If $\Sigma(X)$ is the only connected component, then the surjectivity would follow.

Note that, although $\ker(\rho^H)$ is by definition not visible on $\widetilde{H}(X, \mathbb{Z})$ and by Theorem 4.1 also not on $\mathrm{CH}^*(X)$, it still seems to be governed by the Hodge structure of $\widetilde{H}(X, \mathbb{Z})$. Is this in any way reminiscent of the Bloch conjecture (see 3.3)?

4.5. On the other hand, the image of ρ^H is well understood which is (see [17]):

Theorem 4.3. *The image of $\rho^H : \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{O}(\widetilde{H}(X, \mathbb{Z}))$ is the group $\mathrm{O}_+(\widetilde{H}(X, \mathbb{Z}))$ of all Hodge isometries leaving invariant the natural orientation of the space of positive directions.*

Recall that the Mukai pairing has signature $(4, 20)$. The classes $\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1 - \omega^2/2, \omega$, where $0 \neq \sigma \in H^{2,0}(X)$ and $\omega \in \mathcal{K}_X$ an ample class, span a real subspace V of dimension four which is positive definite with respect to the Mukai pairing. Using orthogonal projection, the orientations of V and $\Phi^H(V)$ can be compared.

To show that $\mathrm{Im}(\rho^H)$ has at most index two in $O(\widetilde{H}(X, \mathbb{Z}))$ uses techniques of Mukai and Orlov and was observed by Hosono, Lian, Oguiso, Yau [14] and Ploog. As it turned out, the difficult part is to prove that the index is exactly two. This was predicted by Szendrői, based on considerations in mirror symmetry, and recently proved in a joint work with Macrì and Stellari [17].

Let us now turn to the image of ρ^{CH}. The only additional structure the Chow group $\mathrm{CH}^*(X)$ seems to have is the subring $R(X) \subset \mathrm{CH}^*(X)$ (see 3.2). And indeed, this subring is preserved under derived equivalences (see [18]):

Theorem 4.4. *If $\rho(X) \geq 2$ and $\Phi \in \mathrm{Aut}(\mathrm{D}^b(X))$, then Φ^H preserves the subring $R(X) \subset \mathrm{CH}^*(X)$.*

In other words, autoequivalences (and in fact equivalences) respect the direct sum decomposition $\mathrm{CH}^*(X) = R(X) \oplus \mathrm{CH}^*(X)_0$ (see 3.2).

The assumption on the Picard rank should eventually be removed, but as for questions concerning potential density of rational points the Picard rank one case is indeed more complicated.

Clearly, the action of Φ^{CH} on $R(X)$ can be completely recovered from the action of Φ^H on $\widetilde{\mathrm{NS}}(X)$. On the other hand, according to the Bloch conjecture (see 3.3) the action of Φ^{CH} on $\mathrm{CH}^*(X)_0$ should be governed by the action of Φ^H on the transcendental part $T(X)$. Note that for $K = \bar{\mathbb{Q}}$ one expects $\mathrm{CH}^*(X)_0 = 0$, so nothing interesting can be expected in this case. However, for $K = \mathbb{C}$ well-known arguments show that $\Phi^H \neq \mathrm{id}$ on $T(X)$ implies $\Phi^{\mathrm{CH}} \neq \mathrm{id}$ on $\mathrm{CH}^*(X)_0$ (see [31]). As usual, it is the converse that is much harder to come by. Let us nevertheless rephrase the Bloch conjecture once more for this case.

Conjecture 4.5. *Suppose $\Phi^H = \mathrm{id}$ on $T(X)$. Then $\Phi^{\mathrm{CH}} = \mathrm{id}$ on $\mathrm{CH}^*(X)_0$.*

By Theorem 4.1 one has $\Phi^{\mathrm{CH}} = \mathrm{id}$ under the stronger assumption $\Phi^H = \mathrm{id}$ not only on $T(X)$ but on all of $\widetilde{H}(X, \mathbb{Z})$. The special case of $\Phi = f_*$ will be discussed in more detail in Section 6

Note that even if the conjecture can be proved we would still not know how to describe the image of ρ^{CH}. It seems, $\mathrm{CH}^*(X)$ has just not enough structure that could be used to determine explicitly which automorphisms are induced by derived equivalences.

5. Chern classes of spherical objects

It has become clear that spherical objects and the associated spherical twists play a central role in the description of $\mathrm{Aut}(\mathrm{D}^b(X))$. Together with automorphisms of X itself and orthogonal transformations of $\widetilde{H}$ coming from universal families of stable bundles, they determine the action of $\mathrm{Aut}(\mathrm{D}^b(X))$ on $\widetilde{H}(X, \mathbb{Z})$. The description of the kernel of ρ^{CH} should only involve squares of spherical twists by Conjecture 4.2.

5.1. It is time to give more examples of spherical objects.

i) Every line bundle $L \in \mathrm{Pic}(X)$ is a spherical object in $\mathrm{D}^{\mathrm{b}}(X)$ with Mukai vector $v = (1, \ell, \ell^2/2 + 1)$ where $\ell = c_1(L)$. Note that the spherical twist T_L has nothing to do with the equivalence given by the tensor product with L. Also the relation between T_L and e.g. T_{L^2} is not obvious.

ii) If $C \subset X$ is a smooth irreducible rational curve, then all $\mathcal{O}_C(i)$ are spherical objects with Mukai vector $v = (0, [C], i + 1)$. The spherical twist $T_{\mathcal{O}_C(-1)}$ induces the reflection $s_{[C]}$ on $\widetilde{H}(X, \mathbb{Z})$, an element of the Weyl group W_X.

iii) Any simple vector bundle E which is also rigid, i.e. $\mathrm{Ext}^1(E, E) = 0$, is spherical. This generalizes i). Note that rigid torsion free sheaves are automatically locally free (see [25]). Let $v = (r, \ell, s) \in \widetilde{N}S(X)$ be a (-2)-class with $r > 0$ and H be a fixed polarization. Then due to a result of Mukai there exists a unique rigid bundle E with $v(E) = v$ which is slope stable with respect to H (see [16]). However, varying H usually leads to (finitely many) different spherical bundles realizing v. They should be considered as non-separated points in the moduli space of simple bundles (on deformations of X). This can be made precise by saying that for two different spherical bundles E_1 and E_2 with $v(E_1) = v(E_2)$ there always exists a non-trivial homomorphism $E_1 \longrightarrow E_2$.

5.2. The Mukai vector $v(E)$ of a spherical object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is an integral $(1, 1)$-class of square -2 and every such class can be lifted to a spherical object. For the Mukai vectors in $\mathrm{CH}^*(X)$ we have the following (see [18]):

Theorem 5.1. *If $\rho(X) \geq 2$ and $E \in \mathrm{D}^{\mathrm{b}}(X)$ is spherical, then $v^{\mathrm{CH}}(E) \in R(X)$.*

In particular, two non-isomorphic spherical bundles realizing the same Mukai vector in $\widetilde{H}(X, \mathbb{Z})$ are also not distinguished by their Mukai vectors in $\mathrm{CH}^*(X)$. Again, the result should hold without the assumption on the Picard group.

This theorem is first proved for spherical bundles by using Lazarsfeld's technique to show that primitive ample curves on K3 surfaces are Brill–Noether general [21] and the Bogomolov–Mumford theorem on the existence of rational curves in ample linear systems [23] (which is also at the core of [4]). Then one uses Theorem 4.1 to generalize this to spherical objects realizing the Mukai vector of a spherical bundle. For this step one observes that knowing the Mukai vector of the Fourier–Mukai kernel of T_E in $\mathrm{CH}^*(X \times X)$ allows one to determine $v^{\mathrm{CH}}(E)$.

Actually Theorem 5.1 is proved first and Theorem 4.4 is a consequence of it, Indeed, if $\Phi : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$ is an equivalence, then for a spherical object $E \in \mathrm{D}^{\mathrm{b}}(X)$ the image $\Phi(E)$ is again spherical. Since $v^{\mathrm{CH}}(\Phi(E)) = \Phi^{\mathrm{CH}}(v^{\mathrm{CH}}(E))$, Theorem 5.1 shows that Φ^{CH} sends Mukai vectors of spherical objects, in particular of line bundles, to classes in $R(X)$. Clearly, $R(X)$ is generated as a group by the $v^{\mathrm{CH}}(L)$ with $L \in \mathrm{Pic}(X)$ which then proves Theorem 4.4.

5.3. The true reason behind Theorem 5.1 and in fact behind most of the results in [4] is the general philosophy that every rigid geometric object on a variety X is already defined over the smallest algebraically closed field of definition of X. This is then combined with the Bloch–Beilinson conjecture which for X defined over $\overline{\mathbb{Q}}$ predicts that $R(X_{\mathbb{C}}) = \mathrm{CH}^*(X)$.

To make this more precise consider a K3 surface X over $\bar{\mathbb{Q}}$ and the associated complex K3 surface $X_{\mathbb{C}}$. An object $E \in \mathrm{D^b}(X_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ if there exists an object $F \in \mathrm{D^b}(X)$ such that its base-change to $X_{\mathbb{C}}$ is isomorphic to E. We write this as $E \simeq F_{\mathbb{C}}$.

The pull-back yields an injection of rings $\mathrm{CH}^*(X) \hookrightarrow \mathrm{CH}^*(X_{\mathbb{C}})$ and if $E \in \mathrm{D^b}(X_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ its Mukai vector $v^{\mathrm{CH}}(E)$ is contained in the image of this map. Now, if we can show that $\mathrm{CH}^*(X) = R(X_{\mathbb{C}})$, then the Mukai vector of every $E \in \mathrm{D^b}(X_{\mathbb{C}})$ defined over $\bar{\mathbb{Q}}$ is contained in $R(X_{\mathbb{C}})$.

Eventually one observes that spherical objects on $X_{\mathbb{C}}$ are defined over $\bar{\mathbb{Q}}$. For line bundles $L \in \mathrm{Pic}(X_{\mathbb{C}})$ this is well-known, i.e. $\mathrm{Pic}(X) \simeq \mathrm{Pic}(X_{\mathbb{C}})$. Indeed, the Picard functor is defined over $\bar{\mathbb{Q}}$ (or in fact over the field of definition of X) and therefore the set of connected components of the Picard scheme does not change under base change. The Picard scheme of a K3 surface is zero-dimensional, a connected component consists of one closed point and, therefore, base change identifies the set of closed points. For the algebraically closed field $\bar{\mathbb{Q}}$ the set of closed points of the Picard scheme of X is the Picard group of X which thus does not get bigger under base change e.g. to $\mathbb{C}$.

For general spherical objects in $\mathrm{D^b}(X_{\mathbb{C}})$ the proof uses results of Inaba and Lieblich (see e.g. [19]) on the representability of the functor of complexes (with vanishing negative Ext's) by an algebraic space. This is technically more involved, but the underlying idea is just the same as for the case of line bundles.

6. Automorphisms acting on the Chow ring

We come back to the question raised as Conjecture 3.4. So suppose $f \in \mathrm{Aut}(X)$ is an automorphism of a complex projective K3 surface X with $f^*\sigma = \sigma$ where σ is a trivializing section of the canonical bundle ω_X. In other words, the Hodge isometry f^* of $H^2(X, \mathbb{Z})$ (or of $\widetilde{H}(X, \mathbb{Z})$) is the identity on $H^{0,2}(X) = \widetilde{H}^{0,2}(X)$ or, equivalently, on the transcendental lattice $T(X)$. What can we say about the action induced by f on $\mathrm{CH}^2(X)$? Obviously, the question makes sense for K3 surfaces defined over other fields, e.g. for $\bar{\mathbb{Q}}$, but $\mathbb{C}$ is the most interesting case (at least in characteristic zero) and for $\bar{\mathbb{Q}}$ the answer should be without any interest due to the Bloch–Beilinson conjecture.

In this section we will explain that the techniques of the earlier sections and of [18] can be combined with results of Kneser on the orthogonal group of lattices to prove Conjecture 3.4 under some additional assumptions on the Picard group of X.

6.1. Suppose $f \in \mathrm{Aut}(X)$ is a non-trivial symplectomorphism, i.e. $f^*\sigma = \sigma$. If f has finite order n, then $n = 2, \ldots, 7$, or 8. This is a result due to Nikulin [28] and follows from the holomorphic fixed point formula (see [26]). Moreover, in this case f has only finitely many fixed points, all isolated, and depending on n the number of fixed point is $8, 6, 4, 4, 2, 3$, resp. 2. The minimal resolution of the quotient $Y \longrightarrow \bar{X} := X/\langle f \rangle$ yields again a K3 surface Y. Thus, for symplectomorphisms of finite order Conjecture 3.4 is equivalent to the bijectivity of the natural map $\mathrm{CH}^2(Y)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^2(X)_{\mathbb{Q}}$. Due to a result of Nikulin the action of a symplectomorphism f of finite order on $H^2(X, \mathbb{Z})$ is as an abstract lattice automorphism independent of f and depends

only on the order. For prime order $2, 3, 5$, and 7 it was explicitly described and studied in [11, 10]. E.g. for a symplectic involution the fixed part in $H^2(X, \mathbb{Z})$ has rank 14. The moduli space of K3 surfaces X endowed with with a symplectic involution is of dimension 11 and the Picard group of the generic member contains $E_8(-2)$ as a primitive sublattice of corank one.

Explicit examples of symplectomorphisms are easy to construct. E.g. $(x_0 : x_1 : x_2 : x_3) \longmapsto (-x_0 : -x_1 : x_2 : x_3)$ defines a symplectic involution on the Fermat quartic $X_0 \subset \mathbb{P}^3$. On an elliptic K3 surface with two sections one can use fibrewise addition to produce symplectomorphisms.

6.2. The orthogonal group of a unimodular lattice Λ has been investigated in detail by Wall in [33]. Subsequently, there have been many attempts to generalize some of his results to non-unimodular lattices. Of course, often new techniques are required in the more general setting and some of the results do not hold any longer.

The article of Kneser [20] turned out to be particularly relevant for our purpose. Before we can state Kneser's result we need to recall a few notions. First, the Witt index of a lattice Λ is the maximal dimension of an isotropic subspace in $\Lambda_{\mathbb{R}}$. So, if Λ is non-degenerate of signature (p, q), then the Witt index is $\min\{p, q\}$. The p-rank $\mathrm{rk}_p(\Lambda)$ of Λ is the maximal rank of a sublattice $\Lambda' \subset \Lambda$ whose discriminant is not divisible by p.

Recall that every orthogonal transformation of the real vector space $\Lambda_{\mathbb{R}}$ can be written as a composition of reflections. The spinor norm of a reflection with respect to a vector $v \in \Lambda_{\mathbb{R}}$ is defined as $-(v, v)/2$ in $\mathbb{R}^*/\mathbb{R}^{*2}$. In particular, a reflection s_δ for a (-2)-class $\delta \in \Lambda$ has trivial spinor norm. The spinor norm for reflections is extended multiplicatively to a homomorphism $\mathrm{O}(\Lambda) \longrightarrow \{\pm 1\}$.

The following is a classical result due to Kneser, motivated by work of Ebeling, which does not seem widely known.

Theorem 6.1. *Let Λ be an even non-degenerate lattice of Witt index at least two such that Λ represents -2. Suppose $\mathrm{rk}_2(\Lambda) \geq 6$ and $\mathrm{rk}_3(\Lambda) \geq 5$. Then every $g \in \mathrm{SO}(\Lambda)$ with $g = \mathrm{id}$ on Λ^*/Λ and trivial spinor norm can be written as a composition of an even number of reflections $\prod s_{\delta_i}$ with (-2)-classes $\delta_i \in \Lambda$.*

By using that a (-2)-reflection has determinant -1 and trivial spinor norm and discriminant, Kneser's result can be rephrased as follows: Under the above conditions on Λ the Weyl group W_Λ of Λ is given by

$$(6.1) \qquad W_\Lambda = \ker \left(\mathrm{O}(\Lambda) \longrightarrow \{\pm 1\} \times \mathrm{O}(\Lambda^*/\Lambda) \right).$$

The assumption on rk_2 and rk_3 can be replaced by assuming that the reduction mod 2 resp. 3 are not of a very particular type. E.g. for $p = 2$ one has to exclude the case $\bar{x}_1 \bar{x}_2$, $\bar{x}_1 \bar{x}_2 + \bar{x}_3^2$, and $\bar{x}_1 \bar{x}_2 + \bar{x}_3 \bar{x}_4 + \bar{x}_5^2$. See [20] or details.

6.3. Kneser's result can never be applied to the Néron–Severi lattice $\mathrm{NS}(X)$ of a K3 surface X, because its Witt index is one. But the extended Néron–Severi lattice $\widetilde{\mathrm{NS}}(X) \simeq \mathrm{NS}(X) \oplus U$

has Witt index two. The conditions on rk_2 and rk_3 for $\widetilde{\mathrm{NS}}(X)$ become $\mathrm{rk}_2(\mathrm{NS}(X)) \geq 4$ and $\mathrm{rk}_3(\mathrm{NS}(X)) \geq 3$. This leads to the main result of this section.

Theorem 6.2. *Suppose* $\mathrm{rk}_2(\mathrm{NS}(X)) \geq 4$ *and* $\mathrm{rk}_3(\mathrm{NS}(X)) \geq 3$. *Then any symplectomorphism* $f \in \mathrm{Aut}(X)$ *acts trivially on* $\mathrm{CH}^2(X)$.

Proof. First note that the discriminant of an orthogonal transformation of a unimodular lattice is always trivial and that the discriminant groups of $\mathrm{NS}(X)$ and $T(X)$ are naturally identified. Since a symplectomorphism acts as id on $T(X)$, its discriminant on $\mathrm{NS}(X)$ is also trivial. Note that a (-2)-reflection s_δ has also trivial discriminant and spinor norm 1. Its determinant is -1.

Let now $\delta_0 := (1, 0, -1)$, which is a class of square $\delta_0^2 = 2$ (and not -2). So the induced reflection s_δ has spinor norm and determinant both equal to -1. Its discriminant is trivial. To a symplectomorphism f we associate the orthogonal transformation g_f as follows. It is f_* if the spinor norm of f_* is 1 and $s_{\delta_0} \circ f_*$ otherwise. Then g_f has trivial spinor norm and trivial discriminant, By Equation (6.1) this shows $g_f \in \widetilde{W}_X$, i.e. f_* resp. $s_{\delta_0} \circ f_*$ is of the form $\prod s_{\delta_i}$ with (-2)-classes δ_i. Writing $\delta_i = v(E_i)$ with spherical E_i allows one to interpret the right hand side as $\prod T_{E_i}^H$.

Clearly, the $T_{E_i}^H$ preserve the orientation of the four positive directions and so does f_*. But s_{δ_0} does not, which proves a posteriori that the spinor norm of f_* must always be trivial, i.e. $g_f = f_*$.

Thus, $f_* = \prod T_{E_i}^H$ and hence we proved that under the assumptions on $\mathrm{NS}(X)$ the action of the symplectomorphism f on $\widetilde{H}(X, \mathbb{Z})$ coincides with the action of the autoequivalence $\Phi := \prod T_{E_i}$. But by Theorem 4.1 their actions then coincide also on $\mathrm{CH}^*(X)$. To conclude, use Theorem 5.1 which shows that the action of Φ on $\mathrm{CH}^2(X)_0$ is trivial. $\qquad\square$

Remark 6.3. The proof actually shows that the image of the subgroup of those $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ acting trivially on $T(X)$ (the 'symplectic equivalences') in $\mathrm{O}(\widetilde{\mathrm{NS}}(X))$ is $\widetilde{W}_X$, i.e. coincides with the image of the subgroup spanned by all spherical twists T_E.

Unfortunately, Theorem 6.2 does not cover the generic case of symplectomorphisms of finite order. E.g. the Néron–Severi group of a generic K3 surface endowed with a symplectic involution is up to index two isomorphic to $\mathbb{Z}\ell \oplus E_8(-2)$ (see [11]). Whatever the square of ℓ is, the extended Néron–Severi lattice $\widetilde{\mathrm{NS}}(X)$ will have $\mathrm{rk}_2 = 2$ and indeed its reduction mod 2 is of the type $\bar{x}_1 \bar{x}_2$ explicitly excluded in Kneser's result and its refinement alluded to above.

Example 6.4. By a result of Morrison [24] one knows that for Picard rank 19 or 20 the Néron–Severi group $\mathrm{NS}(X)$ contains $E_8(-1)^{\oplus 2}$ and hence the assumptions of Theorem 6.2 are satisfied (by far). In particular, our result applies to the members X_t of the Dwork family $\sum x_i^4 + t \prod x_i$ in $\mathbb{P}^3$, so in particular to the Fermat quartic itself. We can conclude that all symplectic automorphisms of X_t act trivially on $\mathrm{CH}^2(X_t)$. For the symplectic automorphisms given by multiplication with roots of unities this was proved by different methods already in [7]. To coe back to the explicit example mentioned before: The involution of the Fermat quartic X_0 given by $(x_0 : x_1 : x_2 : x_3) \longmapsto (-x_0 : -x_1 : x_2 : x_3)$ acts trivially on $\mathrm{CH}^2(X)$.

Although K3 surfaces X with a symplectomorphisms f and a Néron–Severi group satisfying the assumptions of Theorem 6.2 are dense in the moduli space of all (X, f) without any condition on the Néron–Severi group, this is not enough to prove Bloch's conjecture for all (X, f).

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